CAUSTICS IN A TWO-PHASE GAS–PARTICLE MEDIUM

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A continuum-discrete model, in which particles are described by the collisionless kinetic equation, and the gas – by equations of continuous media, has been suggested in [1] for the case of a low bulk particle concentration. It has been shown that condensations in the particle ensemble, called caustics, can occur in this model. The generation and evolution of caustics in a two-phase medium have several features comparable to a medium of noninteracting particles [1, 2], related to particle interactions with the gas. Firstly, caustics are not always generated (in the case of monodisperse particles), but only when the velocity gradient is negative and smaller than some fixed value [1]. Secondly, as shown below, caustics can be generated even when the particle and gas velocities are constant at the moment t = 0, but in this case the particle sizes (or their true density) vary. It is noted that caustics have a different physical nature, unnoticed by the author of [3], since the generation of the latter is due to particle collisions. Vortex sheets are generated at high bulk concentrations, and caustics – at low concentrations. Thus, both models cover the whole region of particle concentrations.

1. We consider one-dimensional nonstationary flow of a mixture of gas with particles with a low particle bulk concentration $\rho_2/\rho_1 \simeq 10^{-2}$. In this case the system of equations of [1] is simplified, since the effect of particles on the gas can be ignored:

$$\frac{\partial f}{\partial t} + u_2 \frac{\partial f}{\partial x} + \frac{\partial}{\partial u_2}\left(\frac{u_1 - u_2}{\tau} f\right) = 0, \quad u_1 = \text{const}, \tag{1.1}$$

$$n = n_0 \int\limits_0^{r_m} dr \int\limits_{-\infty}^{+\infty} f du_2, \quad f = f(t, x, r, u_2),$$

$$m_2 = \pi r^2 \int\limits_{x-r}^{x+r} \left(1 - \left(\frac{x-y}{r_0}\right)^2\right) n(t, y)\, dy, \quad \tau = \frac{2}{9} \frac{\rho_{22}}{\rho_{11}} \frac{r^2}{\nu}.$$

Here ρ_{11}, ρ_1, u_1, ν are the true and mean density, the velocity, and the kinematic viscosity of the gas, ρ_{22}, ρ_2, u_2, m_2, f are the true and mean density, the velocity, the bulk concentration, and the distribution function, and r is the particle radius. The condition $\rho_2/\rho_1 \simeq 10^{-2}$ takes place, for example, for aerosol and water drops in a cloud. The system (1.1) is valid when

$$\text{Kn} = \frac{d}{6 m_2 l_p} \gg 1, \quad l_p \simeq \frac{\rho_{22}}{\rho_{11}} \frac{d}{18} \text{Re}, \quad \text{Re} = \frac{|u_1 - u_2|\, d}{\nu}.$$

Assuming Re ≃ 0.1, $\rho_{22}/\rho_{11} \simeq 10^3$, we obtain Kn ≃ 30 for $m_2 \simeq 10^{-3}$, therefore the system (1.1) is valid up to $m_2 \simeq 10^{-3}$. An equation for m_2 (the particle bulk concentration) was introduced in [4, 5] for flow with large gradients. For smooth flows (z >> r, z being the characteristic size of variation of flow parameters) this equation transforms to the usual $m_2 = (4/3)\pi r^3 n$.

Now consider the Cauchy problem for system (1.1) in the presence of particle dispersion in sizes with the initial conditions

$$f(t = 0) = \frac{1}{\sqrt{2\pi\sigma}} \exp\left(-\frac{(r - r_0)^2}{2\sigma}\right) \delta\left(u_2 - u_2^0(x)\right), \tag{1.2}$$

$$u_2^0(x) = w_s - w \operatorname{arctg} \alpha x, \quad w_s = (w_1 + w_2)/2,$$

$$w = (w_1 - w_2)/\pi, \; w_1 > w_2 > u_1 > 0.$$

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The solution of the first equation of system (1.1) was found in [1] by the method of characteristics. The corresponding integral, determining n(t, x) is calculated by the steepest descent method. Unlike [1], the expansion of the exponent is carried out in the small parameter $\sigma/r_0^2 < 1$. As a result of simple, but quite awkward, calculations we obtain

$$n(t, x) = \sum_{i=1}^{m} n_0/|1 - \beta/(1 + (\alpha x^*)^2)|, \quad \beta = u\alpha K, \quad K = \tau(1 - e^{-t/\tau}), \tag{1.3}$$

$$x^* = x|_{t=0};$$

$$n_{1,2} = n_0 \frac{8^{1/4}\Gamma(1/4)}{4\sqrt{\pi}} \beta^{1/2}\left(1 - 8^{1/4}\frac{\Gamma(3/4)}{\Gamma(1/4)} \times \frac{|x - x_k|}{\theta\sigma^{1/2}}\right)\Big/(\alpha\theta\sigma^{1/2}(\beta - 1)^{1/2})^{1/2}, \tag{1.4}$$

$$|x - x_k| \leqslant \delta_1/4, \quad \delta_1 = \theta\sigma^{1/2}, \quad \alpha\theta\sigma^{1/2} < 1,$$

$$n_{1,2} = n_0 \frac{\beta^{1/2}}{2}\Big/(\alpha|x - x_k|(\beta - 1)^{1/2})^{1/2},$$

$$\delta_1 \leqslant |x - x_k| \leqslant \delta_2, \ \delta_2 = 0.5/\alpha,$$

$$\theta = \frac{2}{9}\frac{\rho_{22}}{\rho_{11}}\mathrm{Re}^0\left(1 - e^{-t/\tau} - \frac{t}{\tau}e^{-t/\tau}\right), \quad \mathrm{Re}^0 = \mathrm{Re}|_{t=0};$$

$$n(t^+) = n_0 \frac{(72)^{1/6}\,\Gamma(1/6)}{6\pi^{1/2}(\alpha\theta\sigma^{1/2})^{2/3}}\left(1 - 19^{2/3}\frac{\Gamma(5/6)}{\Gamma(1/6)}\left(\frac{|x - x_k|}{\theta\sigma^{1/2}}\right)^{2/3}\right), \tag{1.5}$$

$$|x - x_k| \leqslant \frac{1}{4}\delta_1,$$

$$n(t^+, x) = n_0/(3\alpha|x - x_k|)^{2/3}, \ \delta_1 \leqslant |x - x_k| \leqslant \delta_2,$$

where $x_k = x_k(t)$ are the caustic coordinates, and Γ is the gamma function.

The solution (1.3) is valid everywhere, with the exception of a neighborhood of the caustic, determined by the equation $1 - \beta/(1 + (\alpha x^*)^2) = 0$. The equation of the caustic in the t, x plane was found in [1], where was also investigated the behavior of particle trajectories near the caustic.

In Fig. 1 the solid line denotes the caustic, and the dashed line – the particle trajectory [1]. The solution (1.4) describes the dependence n(t, x) near the caustic for $t > t^+$, and Eqs. (1.5) – for $t = t^+$. The corresponding dependences n(x) at three moments of time ($t = 0$, $t = t^+$, $t > t^+$) are shown in Fig. 2 by lines 1-3. We estimate the maximum particle concentration at the caustic, and the width Δ at the half-maximum height of n(x). Putting in (1.4) $\sigma^{1/2}/d \simeq 0.1$, $\mathrm{Re} \simeq 0.1$, $\rho_{22}/\rho_{11} \simeq 10^3$, $\alpha d \simeq 0.1$, $\beta \simeq 2$, we obtain

$$n/n_0 \simeq 4\left(1 - 5.7\frac{|x - x_k|}{d}\right), \quad |x - x_k| \leqslant 0.1d,$$

$$n/n_0 \simeq \sqrt{5}\Big/\left(\frac{|x - x_k|}{d}\right)^{1/2}, \quad d \leqslant |x - x_k| \leqslant 5d,$$

whence $n_{max}/n_0 \simeq 4$, $\Delta \simeq d$.

To determine the bulk particle concentration $m_2(x)$ at the caustic it is necessary to evaluate the integral of n(x) (the third equation in (1.1)). If $\theta\sigma^{1/2}d \gtrsim 1$, from (1.5), (1.1) we have $m_2/m_2^0 \simeq n/n_0$, while if the opposite inequality $\theta\sigma^{1/2}/d \ll 1$ is satisfied, then

$$m_2/m_2^0 \simeq \frac{3}{4}\left(5 - 4\left(\frac{x - x_k}{r_0}\right)^2\right)\Big/(\alpha d)^{2/3} + O(\theta\sigma^{1/2}/d). \tag{1.6}$$

Here $\alpha d \ll 1$; $\left(\frac{x - x_k}{r_0}\right)^2 \ll 1$; and x_k is the coordinate of the point O'.

Figure 3 shows the dependence m_2/m_2^0, calculated from the equation $m_2/m_2^0 = n/n_0$ and Eq. (1.6) for $\alpha d \simeq 0.1$, $\sigma^{1/2}/d \simeq 3\cdot10^{-3}$, $\theta\sigma^{1/2}/d \simeq 6.6\cdot10^{-2}$. It has been shown in [1] that caustics are generated at quite high gradients of $u_2^0(x)$, satisfying the inequality

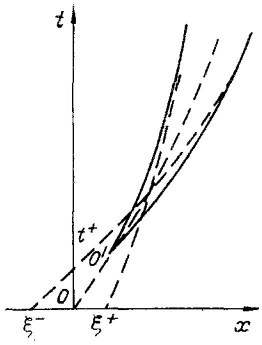

Fig. 1

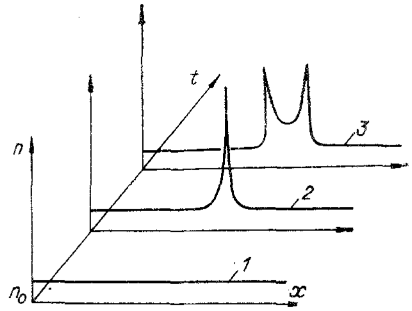

Fig. 2

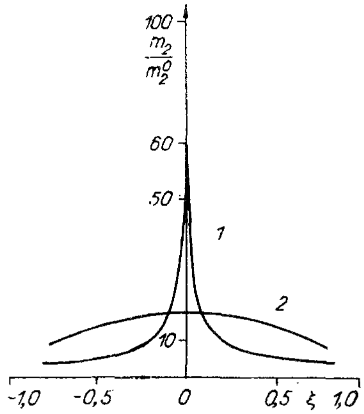

Fig. 3

$$\alpha l_p > 1 \ (l_p \simeq u\tau). \tag{1.7}$$

Taking this condition into account, from (1.6) we find the maximum bulk particle concentration at the caustic $(m_2/m_2{}^0)_{max} \approx 4/(d/\ell_p)^{2/3}$. The boundedness of m_2 at the caustic leads to the circumstance that the number of collisions between particles at the caustic is small, and the initial Knudsen number $N \simeq 1/\mathrm{Kn}^0$ $(\mathrm{Kn}^0 \simeq d/(6m_2^0 l_p) \gg 1)$ is determined. The presence of a critical gradient (1.7) is related to action of the gas on particles: thus, for flows with smaller gradients it is possible to equate the gas and particle velocities up to the generation of caustics.

2. We consider the Cauchy problem for system (1.1) in the case of particle dispersion in velocities with the initial condition

$$f\,(t=0) = \frac{1}{\sqrt{2\pi\sigma}} \exp\left(-\left(u_2 - u_2^0(x)\right)^2/2\sigma\right)\delta\,(r - r_0)$$

($u_2{}^0(x)$ was determined in (1.2)). Carrying out a similar calculation, we obtain outside the caustic

$$n/n_0 = \sum_{i=1}^{m} 1/|1 - \beta/(1 + (\alpha x^*)^2)|, \quad m = 1, 3; \tag{2.1}$$

on the caustic

$$n_{1,2} = n_0 \frac{8^{1/4}\,\Gamma(1/4)}{4\sqrt{\pi}} \frac{\left(1 - 8^{1/4}\frac{\Gamma(3/4)}{\Gamma(1/4)}\frac{\alpha\,|x - x_k|}{\beta\sigma^{1/2}/u}\right)}{((\beta - 1)\,\sigma/u^2)^{1/4}}, \quad 0 \leqslant |x - x_k| \leqslant 0.25\sigma^{1/2}/(u\alpha); \tag{2.2}$$

$$n_{1,2} = 0.5 n_0/((\beta - 1)^{1/2}\alpha\beta\,|x - x_k|)^{1/2}, \quad \sigma^{1/2}/(u\alpha) \leqslant |x - x_k| \leqslant 0{,}5/\alpha, \ \sigma^{1/2}/u < 1; \tag{2.3}$$

and near the point O'

$$n = n_0 \frac{18^{1/6}\Gamma(1/6)}{3\sqrt{2\pi}(\sigma/w^2)^{1/3}}\left(1 - 0.2\frac{\Gamma\left(\frac{5}{6}\right)}{\Gamma\left(\frac{1}{6}\right)}\frac{(\alpha|x - x_k|)^{2/3}}{(\sigma/w^2)^{1/3}}\right). \tag{2.4}$$

$$\begin{gathered} 0 \leqslant |x - x_k| < 0.25\delta_3, \ \delta_3 \simeq 10^{-3}\left(\frac{\sigma}{w^2}\right)^{3/2}\Big/\alpha; \\ n = n_0/(3\alpha|x - x_k|)^{2/3}, \ \delta_3 \leqslant |x - x_k| \leqslant 5\cdot 10^{-2}/\alpha. \end{gathered} \tag{2.5}$$

We estimate the maximum concentration at the caustic and the width for dispersion in velocities. Putting in (2.2) $\beta \simeq 2$, $\alpha d \simeq 0.1$, $\sigma^{1/2}/w \simeq 0.1$, we write

$$\begin{gathered} n/n_0 \simeq 3\left(1 - 0.95\alpha|x - x_k|\Big/\frac{\beta\sigma^{1/2}}{w}\right), \\ 0 \leqslant |x - x_k| \leqslant 0.25d, \quad n/n_0 \simeq \sqrt{5}\Big/\left(\frac{|x - x_k|}{d}\right)^{1/2}, \\ d \leqslant |x - x_k| \leqslant 5d, \end{gathered} \tag{2.6}$$

whence $n_{max}/n_0 \simeq 3$, $\Delta \simeq 2.5d$. The estimates provided show that the condition Kn >> 1 is quite well satisfied at the caustic. This proves the validity of system (1.1) in the given case.

3. Next we consider flow of a mixture of gas with particles of large Reynolds numbers ($Re = |u_1 - u_2|d/\nu \gg 1$). The particle equations of motion are then

$$\frac{dx}{dt} = u_2, \quad \frac{du_2}{dt} = (u_1 - u_2)^2/l_p, \quad l_p \simeq \frac{4}{3}\frac{\rho_{22}}{\rho_{11}}\frac{d}{c_d}, \quad c_d \simeq 0.5. \tag{3.1}$$

We find the solution of the equation at u_1 = const

$$\begin{gathered} u_2 = u_1 - (u_1 - u_2^*)/(1 + t(u_1 - u_2^*)/l_p), \\ x = x^* + u_1 t - l_p \ln\left(1 + \frac{t}{l_p}(u_1 - u_2^*)\right), \quad u_1 > u_2, \\ (u_2^*, x^*) = (u_2, x)|_{t=0}. \end{gathered} \tag{3.2}$$

It is assumed that the particles are located in a channel of constant cross section of length L, that the particle velocity vanishes, the density is constant, and the radius is normally distributed

$$f^* = \frac{1}{\sqrt{2\pi\sigma}}\exp\left(-\frac{(r - r_0)^2}{2\sigma}\right)\delta(u_2^*) \quad (r_0(x^*) = r_s + \Delta r \operatorname{arctg} \alpha x^*). \tag{3.3}$$

The point x = 0 is in the middle of the channel, so that the coordinates of the left and right ends equal, respectively, −L/2, +L/2. At some moment a shock wave, behind whose front the gas moves with velocity u_1, proceeds along the channel toward positive x. The particles start accelerating under the action of the gas, in which case the small ones overcome the larger ones, and a caustic can be generated at some moment. We find the moment of caustic formation and the distance between its branches under the assumption that $tu_1/\ell_p \ll 1$. Expanding in series the right hand side in the second equation (3.2), we obtain

$$x = x^* + \frac{u_1^2 t^2}{2l_p}, \tag{3.4}$$

whence we have with account of (3.3)

$$\frac{\partial x}{\partial x^*} = 1 - \frac{t^2 u_1^2 \varkappa\alpha\Delta r}{2l_p^2(1 + (\alpha x^*)^2)}, \quad \varkappa = \frac{8}{3}\frac{\rho_{22}}{\rho_{11}}\Big/c_d. \tag{3.5}$$

Using (3.4), (3.5), we have the caustic equation

$$x \approx \frac{u_1^2 t^2}{2l_p} \pm \frac{1}{\alpha}\sqrt{(t/t^*)^2 - 1}, \quad t^* = (l_p/u_1)\sqrt{2/(l_p \alpha \Delta r/r_s)}, \tag{3.6}$$

and the distance between caustic branches

$$\Delta x = \frac{2}{\alpha}\sqrt{(t/t^*)^2 - 1}. \tag{3.7}$$

To find the particle concentration at the caustic we consider the identity

$$\int_{-\infty}^{+\infty} f du_2 = \int_{-\infty}^{+\infty} f^* du_2^*, \tag{3.8}$$

where x, t are fixed. As follows from [1], the solution of Eq. (1.1) is

$$f = f^* \exp\left(-\int_0^t \frac{\partial F}{\partial u_2} dt'\right), \tag{3.9}$$

$$\frac{du_2}{dt} = F(u_1 - u_2), \quad \frac{dx}{dt} = u_2, \quad u_1 = \text{const}.$$

Differentiating the second equation of system (3.9) with respect to u_2^*, we write $\frac{d}{dt}\left(\frac{\partial u_2}{\partial u_2^*}\right) = \frac{\partial F}{\partial u_2}\frac{\partial u_2}{\partial u_2^*}$ (t, x are fixed), whence $\frac{\partial u_2}{\partial u_2^*} = \left.\frac{\partial u_2}{\partial u_2^*}\right|_{t=0} \exp\left(\int_0^t \frac{\partial F}{\partial u_2} dt'\right)$. Since $u_2 = u_2^* + \int_0^t F dt'$, under the condition the F be a smooth function, we find

$$\left.\frac{\partial u_2}{\partial u_2^*}\right|_{t=0} = 1, \quad \frac{\partial u_2}{\partial u_2^*} = \exp\left(\int_0^t \frac{\partial F}{\partial u_2} dt'\right). \tag{3.10}$$

Substituting f from (3.9) and $\partial u_2/\partial u_2^*$ from (3.10) into the lefthand side of (3.8), we obtain the required identity.

The identity (3.8) expresses a particle number conservation law, since the left and right hand sides of (3.8) include expressions for the numbers of particles at the same trajectories at the moments t and t = 0. Using identity (3.8) and the initial conditions (3.3) of particle concentration

$$n(t, x) = \frac{n_0}{\sqrt{2\pi\sigma}} \int_{-\infty}^{+\infty} \exp\left(-(r - r_0)^2/2\sigma\right) \frac{\partial r}{\partial x^*} dx^*.$$

The result of evaluating the integral coincides with (1.3)-(1.5), in which case $\beta = (t/t^*)^2$, $\theta = \beta/(\Delta r \alpha)$.

We estimate the characteristic caustic sizes in a channel of length L = 10 cm. Let a bronze particle be found on the left, $\rho_{22} = 8.6$ g/cm^3, $r_0 = 60$ µm, and a bronze particle on the right with $r_0 = 80$ microns whence $r_s = 70$ µm, $\Delta r = 20$ µm, $\ell_p = 84$ cm ($\rho_{11} = 3.8 \cdot 10^{-3}$ g/cm^3).

The transition region from one radius to the other is selected equal to $1/\alpha \simeq 0.21$ cm. A shock wave with M = 3, behind whose front $u_1 = 7.4 \cdot 10^4$ cm/sec, $\rho_{11} = 3.8 \cdot 10^{-3}$ g/cm^3, propagates from left to right. Substituting the corresponding quantities into Eqs. (3.5)-(3.7), we find the caustic formation moment $t^+ = 150$ µsec, $x_k(t^+) = 0.73$ cm, and the distance between caustic branches at the point L/2 $\Delta x \approx 1$ cm. To determine the particle concentration at the caustic we assign $\sigma^{1/2}/d \simeq 0.02$; then, substituting $\beta \simeq 6.8$, $\theta(L/2) \simeq 7 \cdot 10^2$, $\theta(0) \simeq 10^2$, $\alpha d \simeq 0.066$ into (1.4), (1.5), we have $(n/n_0)_k \simeq 1.6$, $n(t^+)/n_0 \simeq 7.26$, and the cluster thickness $\Delta_k \simeq 3.5d$, $\Delta|_{t^+} \simeq 0.5d$, whence $\left(\frac{m_2}{m_2^0}\right)_k \simeq \left(\frac{n}{n_0}\right)_k$. The initial particle concentration is selected from the condition of absence of interparticle collisions in the bulk. Representing the number of particle collisions in the form of sum of collisions at the caustic and outside the caustic $N \simeq \int_0^{0.5} 2\pi d^3 n(x)\, d(x/d) + \pi d^2 n L$, we obtain $N \simeq 6 m_2^0 \left((n/n_0)_{t^+} + L/d\right)$. Hence, taking into account that $(n/n_0)_{t^+} \simeq 7$, $L/d \simeq 10^3$, we find $N \simeq 1/\mathrm{Kn}^0 \simeq 10^{-3}/(6m_2^0)$. Putting Kn ≈ 10, we obtain $m_2^0 \simeq 1.7\cdot 10^{-5}$, $n \simeq 10\ 1/\mathrm{cm}^3$.

4. The results can be generalized to the case of arbitrary coordinate dependence of the initial particle velocity and an arbitrary dependence $c_d(\mathrm{Re}, M_{12})$ with constant gas velocity u_1 = const. For simplicity the particles are assumed to be monodisperse, so that with account of (1.1), (3.8) we write the equation

$$\frac{n}{n_0} = \frac{1}{\sqrt{2\pi\sigma}} \int_{-\infty}^{+\infty} \exp\left(-\frac{\left(u_2^* - u_2^0(x^*)\right)^2}{2\sigma}\right) du_2^*. \quad (4.1)$$

Here t, x are fixed, and $u_2^0(x^*)$ is an arbitrary smooth function.

The integral (4.1) is evaluated by expanding the exponential argument in series $(u_2^* - u_2^+)/\sqrt{\sigma}$ of the function $\varphi^2/2\sigma = \left(u_2^* - u_2^0(x^*)\right)^2/2\sigma$, where $u_2^+ = u_2^0(\xi)$; $x = \xi + \psi(t,\ u_2^0(\xi))$ is the equation of motion. Restricting outselves to the first term in the expansion of φ^2, we obtain

$$n/n_0 = 1\Big/\left|\frac{\partial x}{\partial x^*}\right|. \quad (4.2)$$

Equation (4.2) is valid as long as $\partial x/\partial x^*$ does not vanish. The equation $(\partial x/\partial x^*)_t = 0$ determines the caustic in the t, x* plane, the moment of caustic generation t^+ is found from the condition $(\partial^2 x/\partial x^{*2})_{t^+} = 0$, and the condition of caustic generation is given by the equations $du_2^0/dx^* < 0$, $\left|\frac{du_2^0}{dx^*}\right|\tau > 1$.

To find the solution near the caustic, it is necessary to retain smaller order terms in the expansion of φ^2. As a result of the corresponding calculations we determine the particle concentration at the caustic

$$n/n_0 = 1/(2x_{\xi\xi}\,|x - x_k|)^{1/2},$$
$$(n/n_0)_k = \frac{8^{1/4}\Gamma(1/4)}{2^{3/2}\pi^{1/2}}\Big/\zeta^{1/2},\quad \zeta = \frac{x_{\xi\xi}\sigma^{1/2}}{|du_2^0/d\xi|} \ll 1. \quad (4.3)$$

In the special case $u_2^0(x^*) = w_s - w \arctan \alpha x^*$ and for the Stokes flow regime, the equations obtained (4.3) are transformed to (2.2), (2.3).

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